Scientific project: The Ising Model in three and four dimensions

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Goal of the proposal The proposed collaboration will connect a leading young researcher, of outstanding promise in his cohort, with an established authority in the field. Thematically, the collaboration would link Modern Probability with related Mathematical Physics from which many of its deep challenges originate. It is expected to also form a basis for a network which has room for growth, seeding a collaboration on which a future European Research Council (ERC) grant application (by Hugo Duminil-Copin) may be based for a collaborative program which may grow to involve students and other young researchers, and possibly also other institutions. Princeton and Geneva Universities possess exceptionally strong groups in the respective fields and the collaboration between the two principal investigators could extend to other members of these two research groups in order to tackle related problems.

Scientific motivation Since its formulation by Lenz [Len20], the Ising model has been the most studied example of a system undergoing a phase transition. The model was intended to explain the fact, discovered by Pierre Curie in 1895, that a ferromagnet loses its magnetization when heated above a critical temperature. Beyond its original motivation, the Ising model and its phase transition were found to be of a rather broad relevance, though of course its features do not exhaust the range of possible behaviors in statistical mechanics.

The model also provided the testing ground for a large variety of techniques which have been later used to study other models. Among these techniques, let us mention a few archetypical examples:

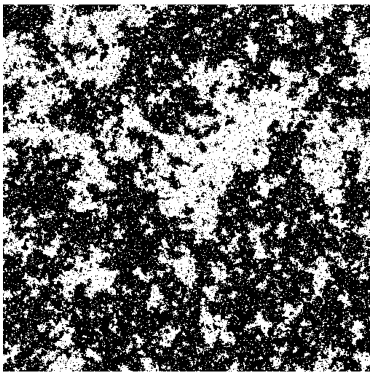
- Transfer matrix methods and their relation to integrability (e.g. Yang-Baxter's equation);
- Differential inequalities relating thermodynamical quantities (such as the spontaneous magnetization or the susceptibility);
- Geometric and Random-walk representations (Fortuin-Kasteleyn percolation, random-currents representation, etc).

Formal definition Associated with the sites of the d-dimensional hypercubic lattice $\mathbb{Z}^d$ are ±1 valued *spin variables*, whose configuration is denoted $\sigma = \{\sigma_x :$

$x \in \mathbb{Z}^d\}$. The *ferromagnetic* Ising model is defined by the family of Hamiltonians:

$$H_\Lambda(\sigma) := - \sum_{\{x,y\} \subset \Lambda : x \sim y} J_{x,y} \sigma_x \sigma_y,$$

for any spin-configuration $\sigma \in \{-1,1\}^\Lambda$, where $(J_{x,y})_{x,y \in \mathbb{Z}^d}$ corresponds to non-negative coupling constants (the model is nearest neighbor if $J_{x,y} = \delta_{\|x-y\|_2 = 1}$).



Current state of research in the field The nearest neighbor ferromagnetic Ising model on $\mathbb{Z}^d$ (for $d = 2$ or $d > 5$) is now well understood. On $\mathbb{Z}^2$, the model is known to be integrable since [Ons44] and conformally invariant since [CS12]. In dimension $d > 4$, the critical exponents were proved to take their mean-field value and the associated quantum field theory to be trivial using the the random-current representation which was introduced by the first PI [Aiz82].

Dimensions 3 and 4 remain the source of great mathematical and physical challenges. In dimension 4, the critical exponents have been computed but the associated quantum field theory has not yet been proved to be trivial. In dimension 3 (which is the dimension of our ambient space), existence of strict power laws remains unproven and so is the existence of the critical model's scaling limit and its relation to 'field theory'. In fact, the phase transition has been proved to be continuous (a prerequisite for the existence of critical exponents) only recently by the two principal investigators [ADCS13]. While much more is known about the critical behavior of this and related models in other dimensions, this is currently the main mathematical result dealing with the critical Ising model in dimension 3.

The goal of this proposal is to capitalize on the recent success (and the techniques developed) in dimension 3 to pursuit the study of the Ising model in dimensions 3 and 4. Among the goals worth pursuing are:

- Develop the random-currents representation which has been the main tool in [ADCS13]
- Prove that the associated quantum field theory is trivial in dimension 4;
- Obtain evidences that the associated quantum field theory is non-trivial in dimension 3;
- Construct the scaling limit of the model in dimension 3.
- Study in more details finite and infinite range models in dimension 2.

References

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